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Strain analysis in rocks with pretectonic fabrics

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Abstract—A major problem in strain analysis of ellipsoidal fabrics is how to separate the effects of initial preferred orientation of objects from the effects of strain. New algebraic methods are presented here which apply to many of the situations previously analysed by graphical and iterative techniques. The algebraic solutions are simple and highlight some assumptions implicit in previous methods, and the relations between those methods. It is shown that many final ellipsoidal fabrics cannot be produced from initial fabrics which were symmetric with respect to bedding, and that a three-dimensional analysis is preferable to analysis on two-dimensional section planes.

INTRODUCTION

MANY methods exist to deduce the strains in deformed rocks by measuring deformed ellipsoidal and elliptical objects (e.g. Cloos 1947, Ramsay 1967, Dunnet 1969, Oertel 1970, Elliott 1970, Dunnet & Siddans 1971, Shimamoto & Ikeda 1976). In early studies the simplest assumption about the pre-deformation shapes of the objects was made, namely that they approximated in shape to spheres. However, the strain ellipsoid deduced by this method does not always have the cleavage as its *XY* symmetry plane. This can be interpreted to mean that the cleavage does not reflect the symmetry of the strain ellipsoid, that markers and matrix did not deform homogeneously, or that the marker objects had an initial preferred orientation. This paper discusses the last possibility.

Elliott (1970) described several types of initial preferred orientation of clasts in sediments. The distributions of these shapes are symmetric with respect to bedding. Oertel (1970) postulated that a bedding-symmetric fabric in an accretionary lapilli tuff could have resulted from compaction of an unconsolidated tuff in which the lapilli had no preferred orientation. Roberts & Siddans (1971) made a similar assumption about pumice shards in ignimbrites. Dunnet & Siddans (1971) assume that the initial fabric of clasts in grits and conglomerates is bedding-symmetric. Thus, the assumption that strain markers in bedded rocks had an initial bedding-symmetric fabric has been applied in a variety of situations.

To deduce the strain in rocks with initial bedding-symmetric fabrics, two main types of approach have been taken. Analysis on two-dimensional section planes has been proposed by Elliott (1970), Siddans (1971), Dunnet & Siddans (1971) and Matthews *et al.* (1974). Alternatively a three-dimensional algebraic approach is taken by Oertel (1970), modified by Bell (1979). These two approaches do not always give comparable results (e.g. Helm & Siddans 1971). In addition to this problem, early methods do not give precisely defined answers: that of Elliott (1970) is graphical, and Oertel (1970) solves his equations by trial-and-error. Later methods attempt to give numerical objectivity but are complex: Dunnet & Siddans (1971) presented an iterative unstraining procedure and Bell (1979) used iteration in two dimensions.

This paper shows that iterative solutions are unnecessary and that there are analytical solutions, best expressed in terms of tensor algebra, for both the 2D and 3D cases. Consideration of the relation between the 2D and 3D approaches shows that there are hidden assumptions in some existing methods, and the new analysis allows the two methods to be reconciled. The new methods use very simple assumptions but give numerical assessments of their validity. In the first part of this paper the methodology is justified for the 2D case, and in the second part it is extended to 3D situations. Finally the 2D and 3D approaches are compared.

TWO-DIMENSIONAL STRAIN ANALYSIS INVOLVING AN INITIAL FABRIC

The 2D case here refers to strain analysis on a section plane. Suppose that the section plane displays a set of deformed elliptical markers, a bedding trace and a cleavage trace. The cleavage trace on the section plane will be parallel to the long axis of the strain ellipse if (1) the section plane is perpendicular to the cleavage (*XY*) plane or (2) the section plane contains a lineation interpreted as the *X* direction. When plotted on an ellipse-shape-orientation grid (e.g. Dunnet 1969, Elliott 1970, Wheeler 1984) the distribution may be asymmetric with respect to cleavage. For instance, on a modified Elliott plot (Wheeler 1984), the centre of gravity of the distribution may not lie on the cleavage trace. This may arise because the markers had an initial preferred orientation of long axes. As discussed above, it is often considered reasonable to assume that the initial distribution of ellipses was symmetric with respect to bedding: their long axes were distributed symmetrically about the bedding trace.

The preferred orientation of a distribution of ellipses can be described by combining the shape and orientation of each ellipse into an average ellipse, which will be referred to as the 'fabric ellipse' (Wheeler in press). In the simplest case when all the ellipses have identical shape and orientation, the fabric ellipse also has this shape and orientation. If the ellipses vary in shape but have a uniform angular distribution of long axes (no preferred orientation) then the fabric ellipse is a circle. In general the fabric ellipse is defined by the 'tensor average' of all the ellipse shapes (Shimamoto & Ikeda 1976, Wheeler in press).

The fabric ellipse itself implies nothing about the strain which the distribution has undergone; it simply summarizes the average properties of the distribution, whether these result from deformation or other processes. Nevertheless it has an important property related to its behaviour when the distribution is deformed: an initial fabric ellipse deforms exactly as if it were a material ellipse subject to the same strain as the distribution. In addition, the fabric ellipse must have the same symmetry as the distribution, or be more symmetric. So, in a bedding-symmetric distribution, the initial fabric ellipse should have the bedding line as a symmetry axis, that is the bedding should be parallel to the major or minor axis. These two properties together make the fabric ellipse a useful tool in strain analysis. Any problem involving the homogeneous deformation of a set of ellipses may be reduced to a problem involving a single ellipse: specifically, the factorization of the final (observed) fabric ellipse into a tectonic strain superimposed on an initial fabric ellipse which satisfied some symmetry criterion. Having simplified the problem in this way, the next step is to ask whether an analytic solution exists for this factorization.

The mathematical formalism behind this is independent of whether the initial fabric was produced by deformation or, for example, by a sedimentary process. If the initial fabric was produced by bedding-symmetric deformation of a random distribution then the problem is one of superimposed strains: for instance in lapilli tuffs, the initial ellipsoidal fabric may be produced by compactional deformation, and the slaty cleavage during later tectonic deformation. Then the final fabric ellipse is the strain ellipse of the total deformation; and the total deformation tensor is the product of the deformation tensors for each stage [Elliott 1972 eqn (5), Ramberg 1975, Schwerdtner & Gapais 1983 eqn (1a), Wheeler 1984 eqn (A1)]. From this may be deduced the relation between the strain ellipse resulting from the first strain and the total strain ellipse resulting from the two superimposed strains as shown in eqn (3) of Appendix 1 [Dunnet 1969 eqn (8a), Shimamoto & Ikeda 1976 eqn (22), Wheeler 1984 eqn (A2)]. If the initial fabric results from some non-tectonic process then it may still be thought of as the result of a 'virtual strain' imposed on a random distribution (Robin 1977). In this way the mathematical treatment of the two cases (tectonic or non-tectonic initial fabric) is seen to be identical. In this paper no assumption is needed about the origin of the initial distribution and so the non-genetic term 'fabric ellipse' is appropriate. Any cleavage or lineation in the rock is supposed to relate to the symmetry of the tectonic strain postdating production of the initial fabric.

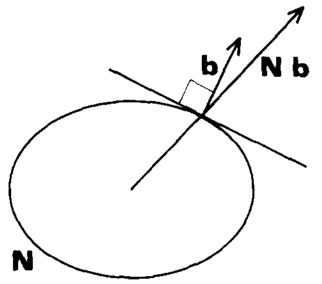

Fig. 1. The geometric significance of **Nb** where **b** is the normal vector to a tangent line (or tangent plane in 3D) and **N** is the ellipse shape tensor. **Nb** joins the ellipse centre to the point where the tangent touches the ellipse.

In Appendix 1 it is shown that an analytical solution for the factorization exists for 2D analysis on a section plane. In this derivation, linear features such as bedding and cleavage traces are described by vectors of unit magnitude perpendicular to the traces. Ellipses including tne strain and fabric ellipses are conveniently described by second-rank 'shape tensors'. These are tensors whose principal vectors are parallel to the ellipse's axes, and whose principal values are the squares of the axial lengths. Thus, the shape tensor for the strain ellipse is the Finger tensor. In general the shape tensor is defined algebraically [Wheeler 1984 eqn (A7)]. In addition to these quantities, the vector **K** is introduced, which may be visualized as follows. Figure 1 illustrates the geometric significance of the vector **Nb** for an ellipse whose ellipse tensor is **N**. Thus **K** is a vector parallel to the line joining the centre of the fabric ellipse to the point where the bedding is tangent to that ellipse (Fig. 2). So eqn (12)

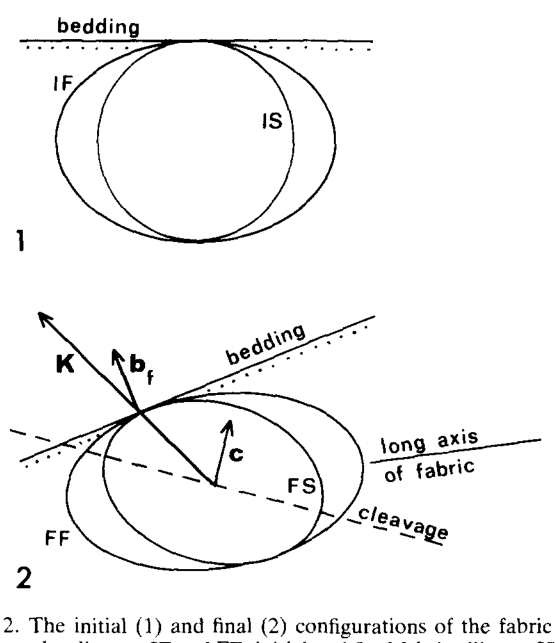

Fig. 2. The initial (1) and final (2) configurations of the fabric in a deformed sediment. IF and FF, initial and final fabric ellipses; IS and FS, initial and final strain ellipses. The initial strain ellipse is a circle.

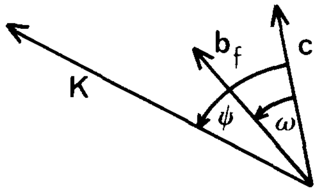

Fig. 3. The meaning of the angles ψ and ω.

is an algebraic expression of the observation that the final fabric and strain ellipses touch where their common tangent is parallel to bedding. As eqn (20) shows, the strain ratio may be expressed in terms of the angles ψ and ω (Fig. 3) or in terms of vector dot products.

An example is given to illustrate the calculation involved in this method. Suppose that we have a section plane on which the cleavage trace pitches at 0°, and the bedding trace pitches at 53° = arctan (4/3). Set up Cartesian coordinates with x parallel to cleavage trace and y pointing down the plane of section. Then the vector **b** is perpendicular to bedding and is given by

$$\mathbf{b} = [-\sin 53°, \cos 53°] = [-4/5, 3/5]$$

and

$$\mathbf{c} = [0, 1].$$

Suppose that the final fabric ellipse has its long axis pitching at 45° and an axial ratio of 2. Then

$$\mathbf{N} = (1/2)\begin{bmatrix} (2 + 1/2) + (2 - 1/2)\cos 90°, & (2 - 1/2)\sin 90° \\ (2 - 1/2)\sin 90°, & (2 + 1/2) - (2 - 1/2)\cos 90° \end{bmatrix}$$

$$= \begin{bmatrix} 5/4, & 3/4 \\ 3/4, & 5/4 \end{bmatrix}$$

To find **K**,

$$\mathbf{K} = \begin{bmatrix} 5/4, & 3/4 \\ 3/4, & 5/4 \end{bmatrix} \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -11/20 \\ 3/20 \end{bmatrix}.$$

from which

$$\tan \psi = 11/3$$

and, knowing

$$\tan \omega = 4/3$$

we find

$$\lambda_1^2 = 11/4.$$

It may be verified that removing this strain ratio from the fabric ellipse and from the bedding line brings them into symmetry.

There are three inbuilt tests of the validity of the assumptions in this method. First, the right-hand side of the expression (19) could be negative, in which case λ_1^2 is imaginary and the problem has no solution. This means that not all ellipse distribution–bedding-cleavage relations can be developed from initial bedding-symmetric distributions. Secondly, λ_1 may be real but less than 1. This would imply that the cleavage was parallel to the short axis of the strain ellipse. Thirdly, w may be greater than 1. Now w relates to the axis of the initial fabric ellipse which is perpendicular to bedding. If $w > 1$, the initial fabric was elongate perpendicular to bedding: the author knows of no recorded situation of such a fabric being observed. The occurrence of any one of these problems would indicate that one or more of the fundamental assumptions was wrong.

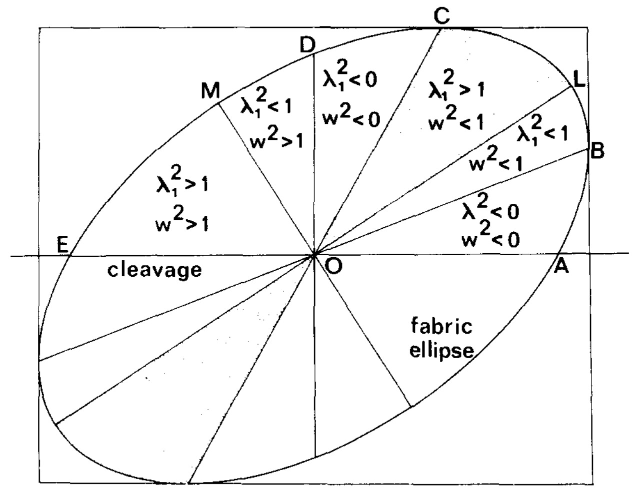

Fig. 4. The relation between cleavage and the fabric ellipse giving rise to a limited field of admissible bedding lines (stippled). See text for explanation.

To understand the limitations imposed upon bedding orientation if the initial fabric was bedding-symmetric, consider Fig. 4. This shows a final fabric ellipse in relation to the cleavage line A–O–E. To unstrain this, a deformation $\mathbf{D}^{-1}$ is applied. This reciprocal finite deformation can always be factorized as follows: (1) a coaxial irrotational deformation with stretching perpendicular to cleavage, but no stretch along cleavage (this does not conserve area); (2) an isotropic area change to give the finite area change as a result of (1) and (2); and (3) a rotation.

Neither the isotropic area change, nor the rotation, will change the relative angular relations between lines in the rock. Therefore we need only consider the effect of step 1, where unstraining consists of moving material points along perpendicular lines away from the cleavage. Material points B and C are where the ellipse touches a rectangle inscribed on the rock, and material point D is where the cleavage normal intersects the ellipse.

The line OL is the long axis of the fabric ellipse and is not a material line. During unstraining, it is confined between material lines OB and OC. So if the bedding line does not lie in this range, it will never coincide with the long axis of the fabric ellipse. Similarly, if it is not in the range OD to OE it will never coincide with the short axis. This type of argument can be extended to label all the regions of the ellipse according to what λ_1^2 and w^2 will be like, given a bedding line in that region. As discussed above, the only geologically realistic solutions are those with $\lambda_1^2 > 1$ and $w^2 < 1$. This confines the bedding line to the narrow sector between OL and OC.

If the final bedding is parallel or perpendicular to the cleavage, infinite or indeterminate strains are yielded by

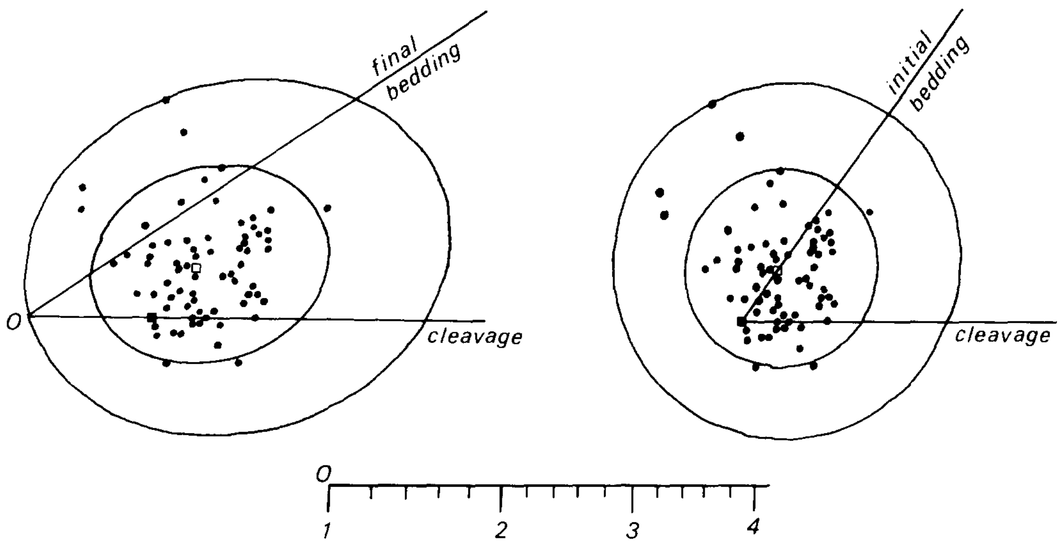

Fig. 5. Data from Dunnet & Siddans (1971) plotted on a modified Elliott plot (Wheeler 1984). 'O' marks the origin of the plot, and the scale gives ellipse axial ratios as a function of distance from the origin. The angle an ellipse long axis makes with a reference line is doubled and used as angular coordinate on the plot. The left-hand diagram illustrates a measured ellipse distribution which is asymmetric to bedding. The open square marks the fabric ellipse; the filled square the strain ellipse as calculated from eqn (19). The two 'onion curves' are the loci of all those ellipses having pre-compaction axial ratios of 1.5 and 2.0 (the fabric ellipse has an axial ratio of *c*. 2 and therefore the R = 2 onion goes through the plot origin). The unstrained distribution (right) shows the initial fabric ellipse lying on the initial bedding line. The initial bedding-parallel fabric is assumed, in this accretionary lapilli tuff, to be due to compaction.

(19). This agrees with other work: because the bedding is unaltered by removing strain, the marker ellipses can never be brought into symmetry with bedding. The only exception to this is when the final fabric is still parallel to bedding. This case will be discussed later.

Figure 5 shows an application of the new method to the data of Dunnet & Siddans (1971 fig. 11B). The strain ratio is 1.67 and the initial fabric ratio is 1.30; both values are in good agreement with their estimates.

The applicability of two-dimensional strain analysis

Dunnet & Siddans (1971) used a 2D method of strain analysis (formalized as the program STRANE) on each of three section planes, and deduced the strain ellipsoid by combining the three sectional strain ellipses. The 2D method minimizes the difference between the bedding trace and the mean long axis of the 2D ellipse distribution. There is little conceptual difference between the mean long axis, and the long axis of the mean (i.e. fabric) ellipse. For this reason the assumptions on which the 2D method of this paper are based are similar to those of Dunnet & Siddans (1971). These are (1) initially the bedding trace was parallel to the long axis of the section through the initial fabric ellipsoid and (2) the cleavage trace is parallel to the long axis of the section ellipse through the strain ellipsoid.

To consider the effect of sectioning on symmetry, note that a triaxial ellipsoid is symmetric with respect to, for example, its XY plane, while the section ellipse on a plane oblique to all axes is not symmetric to the XY plane trace. Symmetry only occurs when the section plane contains one or more of the principal axes of the ellipsoid, or on every section plane if the ellipsoid is uniaxial.

Dunnet & Siddans (1971) use their method on section planes parallel or perpendicular to the cleavage and this is valid even if the fabric ellipsoid is triaxial. In addition the initial fabric ellipsoid must be assumed to be uniaxial, since section planes are not in general in any special orientation relative to the initial fabric. It is this assumption which can be tested by devising a fully 3D algebraic method and comparing the 2D method with it.

THREE-DIMENSIONAL STRAIN ANALYSIS INVOLVING INITIAL FABRIC

It is possible to classify strain marker fabrics on the basis of their symmetry to bedding. Dunnet & Siddans (1971) recognized three types of strain marker fabrics of which one, the 'imbricate' fabric, is asymmetric to bedding and is not discussed further. The 'planar' and 'random-imbricate' fabrics are symmetric to bedding and also have a rotational symmetry about the bedding normal. As in the two-dimensional case, the fabric ellipsoid must have the symmetry of the distribution, and so must be uniaxial with the bedding normal as unique axis. Similarly, the initial fabric resulting from compaction of initial random sedimentary fabrics (e.g. accretionary lapilli, see Oertel 1970) will also be uniaxial (Sanderson 1976).

The only basic criterion for the following analysis is that the initial fabric ellipsoid is assumed to have the bedding normal as a principal axis. So it can deal with planar, random-imbricate and compactional fabrics, and also with 'triaxial-symmetric' fabrics. A triaxial-symmetric fabric is defined as follows: any distribution which is symmetric with respect to bedding but where the ellipsoids have a preferred orientation within bedding. The author knows of no reference to such a fabric, and they are unlikely to be produced in clasts by sedimentary processes. However, Roberts & Siddans (1971) deduce a triaxial 'welding deformation ellipsoid' for pumice shards in an ignimbrite. They interpret this as due to a component of downhill flow and extension during compaction. Triaxial-symmetric fabrics are important simply because they are often predicted (sometimes implicitly) by many existing methods of strain analysis.

As in the 2D case, the first step in the analysis is the determination of the fabric ellipsoid. This can be done by determining a best-fit ellipsoid to the elliptical sections measurable on section-planes (e.g. Owens 1984). Alternative methods are discussed by Wheeler (in press). In the following discussion it will be assumed that the fabric ellipsoid has been determined. It must then be factorized into an initial fabric and a superimposed strain.

STRAIN ANALYSIS OF ROCKS WITH CLEAVAGE AND LINEATION

In deformed lapillar tuffs from the Lake District, a lineation is present on the cleavage surfaces which is interpreted as indicating the long axis of the finite strain ellipsoid (Oertel 1970, Bell 1979). So the cleavage and lineation together define the orientation of the strain ellipsoid. In Appendix 2 it is shown that this information is sufficient to define the state of strain in the rock. Equations (27) and (28) are the analytic solution to the problem which has been treated by trial and error (Oertel 1970) and by a complex procedure involving iteration with two variables (Bell 1979).

As in the 2D case, solutions are only geologically valid if $w > 0$ and $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Consideration of eqns (24)–(26) shows that this constrains $\mathbf{b}_f$ and $\mathbf{K}$ to lie in the same octant of the strain ellipsoid; i.e. they must not be separated by a principal plane of that ellipsoid. It can be seen that the solution to the problem is exactly specified by known bedding, cleavage, lineation and final fabric ellipsoid. It is important to note that the initial fabric ellipsoids deduced by this method will in general be triaxial-symmetric.

STRAIN ANALYSIS IN ROCKS WITH CLEAVAGE BUT NO LINEATION

Some authors (e.g. Dunnet & Siddans 1971) have produced results from rocks with bedding and cleavage but no lineation. Since the problem is only just specified when there is a known lineation, it is underspecified when the lineation is unknown. To deduce the strain, some additional assumption must be made. As discussed above, many initial bedding-symmetric fabrics are uniaxial. This assumption can be incorporated into the method of strain analysis, following the example of Oertel (1970) and Dunnet & Siddans (1971). With the uniaxial assumption we can see whether an internally consistent solution exists. The shape of a triaxial ellipsoid is specified by five pieces of information (two axial ratios, the trend and plunge of one axis, and the trend of another axis). A uniaxial ellipsoid shape only needs three pieces of information (one axial ratio and the trend and plunge of the unique axis). Compared to the problem in the previous section, the absence of lineation implies one less item of data, but the assumption that the initial fabric ellipsoid was uniaxial reduces the number of unknowns by two. Therefore the problem is overspecified.

The uniaxial assumption must therefore force a constraint on the known data: there must be a relationship between bedding, cleavage and final fabric ellipsoid. Even if the assumption is correct, the real data will suffer from measurement errors and will never satisfy the constraint exactly. The degree to which it departs from the constraint will reflect the validity of the uniaxial assumption. The assumption of initial uniaxial fabric is comparable to an assumption that the strain ellipsoid is uniaxial; either one of these might be appropriate. The assumption that the strain ellipsoid is uniaxial is slightly simpler algebraically and is discussed first.

Strain analysis assuming a uniaxial strain ellipsoid

A uniaxial strain ellipsoid of oblate ($k = 0$) type does not produce a unique stretching direction in the cleavage plane, and it is plausible that this can account for the absence of lineation in the situation under discussion. In Appendix 3, the shape of the strain ellipsoid is defined by deriving the value of the quadratic stretch in the cleavage plane [eqn (39)]. There are two features of the derivation which deserve comment. First, the expression (40) for the quadratic stretch is analogous to (20) because λ_1^3 is the square of the axial ratio of the strain ellipsoid. This, then, is an example of a case in which 2D strain analysis will work: it works in the plane which contains $\mathbf{c}$, $\mathbf{b}_f$ and $\mathbf{K}$. To see why, consider the behaviour of bedding under a uniaxial strain. During strain, the pole to bedding moves on a great circle towards the pole to cleavage. Therefore, $\mathbf{b}_i$, $\mathbf{b}_f$ and $\mathbf{c}$ must be coplanar. Hence the plane containing $\mathbf{c}$, $\mathbf{b}_f$ and $\mathbf{K}$ also contains $\mathbf{b}_i$. Because it contains $\mathbf{b}_i$, this plane will show an initial fabric ellipse symmetric to the bedding trace, even if the initial fabric is triaxial. Thus, the 2D method will work on this plane.

The second feature concerns how the constraint on the data manifests itself: it emerges when it is shown that the three vectors $\mathbf{b}_f$, $\mathbf{c}$ and $\mathbf{K}$ should be coplanar. Since all these vectors are measurable and therefore already specified in the problem, this coplanarity is a constraint on the input data. The occurrence of such a constraint has already been predicted. A measure of the departure from coplanarity is given by μ [eqn (35)]. The analytical solution is then realistic if μ is near zero and $\lambda_1 > 1$. In practice because μ is different from zero, the input data must be modified to satisfy the constraint exactly, before any other calculations are done. The justification for this 'modification' will be discussed later. Out of all the pieces of input data, it is the final fabric ellipsoid which is subject to the most measurement error, and is likely to be responsible for the deviation from the constraint if that deviation is statistical in origin. This suggests one should correct $\mathbf{N}_f^*$, and a method of doing this is given in eqn (42).

Strain analysis assuming a uniaxial initial fabric ellipsoid

As discussed by Oertel (1970) and Sanderson (1976), many compactional fabrics can be expected to have a uniaxial oblate fabric ellipsoid, since in simple cases compaction is constrained in all horizontal directions within bedding. Derivation of a formula for u, the squared length of the bedding-parallel radius of the initial fabric ellipsoid, is given in Appendix 4. Knowing u, the Finger tensor is given by eqn (58) and this defines the state of strain. As before the constraint on the data is manifested as the coplanarity of three vectors, and departure from this condition is measured by ν.

DISCUSSION

Table 1 shows an example of the application of each of the new methods to measurements of deformed conglomerates in the Mellene Nappe, Norway. All calculations are done by an Algol68 program which is available on request. It should be borne in mind that no statement can be made about the rotational component of the deformation, apart from the fact that the bedding was presumably initially horizontal. It is not clear from this example alone which of the assumptions about strain was valid, if any. The assumption that the initial fabric was uniaxial seems intuitively reasonable, but predicts a finite stretching direction which is not close to the measured lineation. Conversely, assuming the lineation represents X leads to the deduction of a triaxial-symmetric initial fabric.

An important feature of the new methods is the introduction of constraints on the input data. In previous methods such constraints are present but are not stated explicitly. For example the method of Dunnet & Siddans (1971) for analysis on 2D section planes assumes the initial fabric is uniaxial (but it has been applied to deduce triaxial-symmetric initial fabrics, e.g. Roberts & Siddans 1971). Having deduced a strain ellipse for each section plane, these are combined to form the strain ellipsoid. There may be some incompatibility between the strain ellipses. In simple situations this incompatibility is assumed to be statistical in origin (Milton 1980). It has been shown above that not all fabrics can be unstrained to uniaxial initial fabrics. Therefore the 'incompatibility' that 2D methods can produce is a combination of statistical effects and the effect of constraints such as (36). To fit a 3D ellipsoid to incompatible 2D section ellipses, each section ellipse must be corrected. In the new methods described here the correction is made explicitly

Table 1. Results of the application of three different methods of strain analysis to data from a deformed conglomerate

Final fabric ellipsoid						
Axial lengths	0.654	1.104	1.386			
Axial directions	42/182	43/331	16/077			
Bedding 260/55	Cleavage 308/38		Lineation 32/074			
Assume lineation is finite tectonic extension direction						
	Strain ellipsoid			Initial fabric ellipsoid		
Axial lengths	0.724	1.072	1.288	0.772	1.014	1.278
Axial directions	52/218	18/332	32/074	29/158	47/032	29/265
Initial bedding 247/61						
Assume strain ellipsoid was uniaxial						
Constraint $\mu = 0.0785$						
Corrected final fabric ellipsoid						
Axial lengths	0.651	1.111	1.383			
Axial directions	45/181	41/332	15/075			
	Strain ellipsoid		Initial fabric ellipsoid			
Axial lengths	0.754	1.152	0.741	1.036	1.302	
Axial directions	52/218	—	27/160	61/004	10/255	
Initial bedding 250/63						
Assume uniaxial initial fabric						
Constraint $\nu = -0.0981$						
Corrected final fabric ellipsoid						
Axial lengths	0.655	1.124	1.359			
Axial directions	41/183	43/327	19/076			
	Strain ellipsoid			Initial fabric ellipsoid		
Axial lengths	0.776	1.014	1.272	0.724	1.175	
Axial directions	52/218	32/002	18/104	30/156	—	
Initial bedding 246/60						

and at the beginning of the calculation. It has nothing to do with statistical incompatibility. The effects of this should be considered when measuring the final fabric ellipsoid, before any unstraining is done.

As a second example, Oertel (1970) discussed a 3D approach to a situation where both cleavage and lineation were known. It was recognized that to unstrain a final fabric ellipsoid to an initial uniaxial fabric ellipsoid, there must be constraints on the known data. If the constraints are satisfied within measurement error, this is evidence in favour of the uniaxial hypothesis. Oertel presents a 'calculated' final fabric which differs from the 'observed' final fabric. This is another example of the correction of a final fabric to fit with a model, and confirms that such corrections are intrinsic to the analysis of rocks with pretectonic fabrics.

Any method of strain analysis using angular relations between fabric, bedding, etc. breaks down when bedding and cleavage are parallel or perpendicular, or have some symmetry relation to the fabric ellipsoid. Such methods become less precise when final geometries are close to having some symmetry. In such cases estimates of strain can be made by estimating the actual axial lengths of the initial fabric ellipsoid. For instance, in the 2D case we may find the final fabric is parallel to bedding and cleavage is parallel or perpendicular to it. An estimate of the axial ratio of the initial fabric ellipse allows the strain to be deduced. In 3D, the method of Owens (1974) can be used when the final situation is highly symmetric.

The usefulness of 2D methods can now be discussed. It has been pointed out that 2D methods cannot be used when the initial fabric was triaxial-symmetric, whereas a 3D method (the case where the lineation is known) can give an answer. Conversely, if the initial fabric is assumed uniaxial, 2D methods can be applied, but only after the fabric ellipsoid has been adjusted to satisfy the essential constraint. Since calculating that correction involves a fully 3D analysis, there seems little point in returning to the section planes to calculate the strain. In addition, some authors (e.g. Bell 1979) have reported symmetric ellipse distributions on some section planes, when other section planes indicate clearly that the ellipse distribution is asymmetric. This is a result of unfortunate choice of section planes and means that 2D methods cannot be used although a 3D method will yield a result.

The importance of section plane data is in its potential to be displayed graphically. Although the strain ellipsoid should be calculated by 3D methods, it is useful to unstrain all the section plane data and examine the appearance of the initial distribution. It should be borne in mind that the initial distribution may look asymmetric to the bedding trace on a section plane for several distinct reasons.

(1) The sectional shape of the fabric ellipsoid is not necessarily quite the same as the average ellipse shape on a section plane (Wheeler in press).

(2) The fabric ellipsoid may be 'corrected' to accord with constraints. However, it is not clear how the raw section plane data should be altered to accord with this. A discrepancy is inevitable when the strain has been derived from a modified fabric ellipsoid, whereas the raw data is unmodified.

(3) If the initial fabric is triaxial-symmetric, the bedding trace need not be symmetric to the fabric ellipse.

(4) The distribution may be asymmetric in some other way even if its centroid lies on the bedding trace. An example of such an asymmetry is expressed by the ISYM3 parameter of Dunnet & Siddans (1971).

CONCLUSIONS

It has been shown that it is possible to analyse the deformation of rocks with pretectonic ellipsoid fabrics by considering the fabric ellipsoid (average ellipsoid shape) and attempting to factorize the final fabric ellipsoid into a tectonic strain superimposed on an initial fabric ellipsoid which has bedding as a symmetry plane. To do this, some assumptions must be made. The following alternatives have been analysed.

(1) The lineation is parallel to the long axis of the strain ellipsoid.

(2) The strain ellipsoid is uniaxial.

(3) The initial fabric ellipsoid was uniaxial.

An algebraic solution to the factorization can be derived in each of these situations. An algebraic solution also exists to the 2D problem in which the bedding, cleavage and final fabric are known. However, section planes can often show asymmetry even when the 3D situation is symmetric, so a 3D algebraic solution is always advisable. These algebraic methods are equivalent to published iterative procedures, but are precise and allow the relation between the various approaches to be considered.

The new methods demonstrate when a factorization is not possible, and also show up constraints which must apply to the measurable data when either of the uniaxial assumptions are made. The magnitude of deviation from the constraint is expressed numerically by the values μ and ν, and these can be used on a set of factorizations to decide which of the assumptions is in general the most appropriate.

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Table 2. Nomenclature

$\mathbf{x}$	position vector
$\mathbf{b}$	bedding normal unit vector
$\mathbf{c}$	cleavage normal unit vector
$\mathbf{l}$	unit vector parallel to lineation
$\mathbf{q}$	unit vector orthogonal to $\mathbf{c}$ and $\mathbf{l}$
$\mathbf{m}$	unit vector parallel to bedding-cleavage intersection
$\mathbf{g}$	unit tensor
$\mathbf{D}$	deformation tensor
$\mathbf{N}^*$	fabric ellipsoid tensor
$\mathbf{N}_s$	strain ellipsoid tensor (Finger tensor)
$\lambda_1, \lambda_2, \lambda_3$	quadratic stretches of strain ellipsoid
u, v, w	squared lengths of axes of initial fabric ellipsoid
$\mathbf{K}$	vector parallel to line joining centre of final fabric ellipsoid to point where bedding plane is tangent to it
$\mathbf{Q}$	vector parallel to line joining centre of final fabric ellipsoid to point where cleavage is tangent to it
$\mathbf{v}$	unit vector parallel to $\mathbf{K}$
$\mathbf{t}$	unit vector parallel to $\mathbf{Q}$
ω	angle between $\mathbf{b}_f$ and $\mathbf{c}$
ψ	angle between $\mathbf{K}$ and $\mathbf{c}$
μ, ν	measures of deviation from constraints
i, f	subscripts denoting initial, final.

APPENDIX 1: 2D STRAIN ANALYSIS ON A SECTION PLANE

The fabric ellipse tensor is a symmetric tensor of unit determinant which has its eigenvectors parallel to the ellipse's axes, and its eigenvalues are the squares of the lengths of the axes. If $\mathbf{N}^*$ is the fabric ellipse tensor, the equation of the fabric ellipse would be

$$\mathbf{x} \cdot (\mathbf{N}^*)^{-1}\mathbf{x} = 1. \tag{1}$$

The bedding line will be represented in terms of the unit vector normal to it (this is so equations are easily extended to 3D). The normal vector $\mathbf{b}$ has directional ambiguity (it can point either up or down) but, once it is chosen, this does not affect the argument. Let the subscripts i and f refer to initial and final quantities (before and after deformation). Then the bedding-symmetric condition is written as

$$\mathbf{N}_i^*\mathbf{b}_i = w\mathbf{b}_i, \tag{2}$$

where w is an eigenvalue of the initial fabric ellipse. The axis of the fabric ellipse perpendicular to bedding has length $\sqrt{w}$.

Consider a deformation whose deformation tensor (position gradients tensor) is $\mathbf{D}$. It may be rotational and it may have resulted from a non-coaxial strain history, but the calculation is the same in any case. For simplicity $\mathbf{D}$ will be considered to have unit determinant, so there is no volume change. This avoids carrying det $\mathbf{D}$ through all the algebra. Nevertheless the method to be described uses only angular relations and still gives the shape of the strain ellipsoid even when there is volume change. Deformation affects the fabric ellipse tensor as if it were a material ellipse tensor.

$$\mathbf{N}_f^* = \mathbf{D}\mathbf{N}_i^*\mathbf{D}^T \tag{3}$$

and the bedding normal deforms according to

$$\mathbf{b}_f = (\mathbf{D}^T)^{-1}\mathbf{b}_i(\mathbf{b}_i \cdot (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{b}_i)^{-1/2} \tag{4}$$

(see for instance Owens 1973). The last term in (4) is simply to rescale $\mathbf{b}$ so it has unit length. These equations can be recast to give the initial quantities in terms of the final quantities.

$$\mathbf{N}_i^* = \mathbf{D}^{-1}\mathbf{N}_f^*(\mathbf{D}^T)^{-1} \tag{5}$$

and

$$\mathbf{b}_i = \mathbf{D}^T\mathbf{b}_f(\mathbf{b}_f \cdot (\mathbf{D}\mathbf{D}^T)\mathbf{b}_f)^{-1/2}. \tag{6}$$

Substitute (5) and (6) into (2): because the normalization term for $\mathbf{b}$ appears on both sides, it can be dropped to give

$$\mathbf{D}^{-1}\mathbf{N}_f^*(\mathbf{D}^T)^{-1}\mathbf{D}^T\mathbf{b}_f = w\mathbf{D}^T\mathbf{b}_f. \tag{7}$$

Dot multiply both sides with a preceding $\mathbf{D}$ and simplify to give

$$\mathbf{N}_f^*\mathbf{b}_f = w\mathbf{D}\mathbf{D}^T\mathbf{b}_f. \tag{8}$$

Now the Finger tensor occurs in the expression on the right-hand side of the equation, and the Finger tensor is the ellipse tensor (formally defined in Wheeler in press) of the strain ellipse. So, if $\mathbf{N}_s$ is the strain ellipse tensor,

$$\mathbf{D}\mathbf{D}^T = \mathbf{N}_s. \tag{9}$$

Substitute this in (8) to find

$$\mathbf{N}_f^*\mathbf{b}_f = w\mathbf{N}_s\mathbf{b}_f. \tag{10}$$

This is a simple formula in which the left-hand side is entirely in terms of known quantities; so define

$$\mathbf{K} = \mathbf{N}_f^*\mathbf{b}_f \tag{11}$$

so that

$$w\mathbf{N}_s\mathbf{b}_f = \mathbf{K}. \tag{12}$$

The only unknowns in this equation are w and $\mathbf{N}_s$ (which is specified by an axial ratio and an angle). So (12) represents two equations in three unknowns. To obtain the strain ellipse we thus need one further piece of information. This is provided by the cleavage line, which is parallel to the long axis of the strain ellipse on any section plane perpendicular to cleavage.

Let $\mathbf{c}$ be the unit normal vector to the cleavage line, and $\mathbf{l}$ be a unit vector perpendicular to $\mathbf{c}$ and along the cleavage line. Let the quadratic stretch along the cleavage be λ_1, then because the deformation is taken as area-conserving for simplicity, the quadratic stretch along $\mathbf{c}$ is $1/\lambda_1$. So:

$$\mathbf{N}_s\mathbf{l} = \lambda_1\mathbf{l} \tag{13}$$

$$\mathbf{N}_s\mathbf{c} = (1/\lambda_1)\mathbf{c}. \tag{14}$$

Consider dot-multiplying both sides of (11) with $\mathbf{l}$. Then

$$w\mathbf{l} \cdot (\mathbf{N}_s\mathbf{b}_f) = \mathbf{l} \cdot \mathbf{K} \tag{15}$$

but the dot products are associative and $\mathbf{N}$ is symmetrical so

$$w(\mathbf{N}_s\mathbf{l}) \cdot \mathbf{b}_f = \mathbf{l} \cdot \mathbf{K} \tag{16}$$

and, applying (13),

$$w\lambda_1 \mathbf{l}\cdot\mathbf{b}_f = \mathbf{l}\cdot\mathbf{K}. \quad (17)$$

A similar procedure using **c** yields

$$(w/\lambda_1)\mathbf{c}\cdot\mathbf{b}_f = \mathbf{c}\cdot\mathbf{K} \quad (18)$$

(17) and (18) are two equations in two unknowns, λ_1 and w. Solving for λ_1 gives:

$$\lambda_1^2 = \frac{(\mathbf{l}\cdot\mathbf{K})(\mathbf{c}\cdot\mathbf{b}_f)}{(\mathbf{c}\cdot\mathbf{K})(\mathbf{l}\cdot\mathbf{b}_f)}. \quad (19)$$

Note that λ_1 is the axial ratio of the strain ellipse as well as the quadratic stretch along **l**. It is straightforward to reexpress this in the cleavage coordinate frame. Let ψ be the angle between **K** and **c**, and ω be the angle between $\mathbf{b}_f$ and **c**. Then (19) becomes

$$\lambda_1^2 = \frac{\tan\psi}{\tan\omega}. \quad (20)$$

APPENDIX 2: 3D STRAIN ANALYSIS IN ROCKS WITH CLEAVAGE AND LINEATION

To analyse the situation algebraically, note that the derivation of equation (12) can be carried over without alteration to the 3D case. Let λ_1, λ_2, λ_3 be the quadratic stretches of the strain ellipsoid. Let **c** be the unit vector perpendicular to cleavage and parallel to Z; **l** be parallel to the lineation and parallel to X; and **q** be the unit vector perpendicular to both these and hence parallel to Y. Then, similarly to (13) and (14), we can write

$$\mathbf{N}_s\mathbf{l} = \lambda_1\mathbf{l} \quad (21)$$

$$\mathbf{N}_s\mathbf{q} = \lambda_2\mathbf{q} \quad (22)$$

$$\mathbf{N}_s\mathbf{c} = \lambda_3\mathbf{c} \quad (23)$$

Dotting (12) with **l**, **q** and **c** and rearranging as in the 2D case gives:

$$w\lambda_1\mathbf{l}\cdot\mathbf{b}_f = \mathbf{l}\cdot\mathbf{K} \quad (24)$$

$$w\lambda_2\mathbf{q}\cdot\mathbf{b}_f = \mathbf{q}\cdot\mathbf{K} \quad (25)$$

$$w\lambda_3\mathbf{c}\cdot\mathbf{b}_f = \mathbf{c}\cdot\mathbf{K} \quad (26)$$

Because $\det \mathbf{N}_s = 1$, we have $\lambda_1\lambda_2\lambda_3 = 1$ and the three equations can be solved for w:

$$w^3 = \frac{(\mathbf{l}\cdot\mathbf{K})(\mathbf{q}\cdot\mathbf{K})(\mathbf{c}\cdot\mathbf{K})}{(\mathbf{l}\cdot\mathbf{b}_f)(\mathbf{q}\cdot\mathbf{b}_f)(\mathbf{c}\cdot\mathbf{b}_f)} \quad (27)$$

and so

$$\lambda_1 = \frac{\mathbf{l}\cdot\mathbf{K}}{w\mathbf{l}\cdot\mathbf{b}_f},\ \lambda_2 = \frac{\mathbf{q}\cdot\mathbf{K}}{w\mathbf{q}\cdot\mathbf{b}_f},\ \lambda_3 = \frac{\mathbf{c}\cdot\mathbf{K}}{w\mathbf{c}\cdot\mathbf{b}_f}. \quad (28)$$

Knowing these quadratic stretches, the corresponding irrotational deformation can be calculated, and this substituted in (5) and (6) to derive the initial fabric and bedding, specified apart from an unknown rotation.

APPENDIX 3: 3D STRAIN ANALYSIS ASSUMING A UNIAXIAL STRAIN ELLIPSOID

As the strain ellipsoid is uniaxial, for any vector in the cleavage plane, the quadratic stretch will be λ_1 $(= \lambda_2)$. For vectors perpendicular to it the quadratic stretch will be $\lambda_3 = 1/\lambda_1^2$. Consider a coordinate frame in which **c** is parallel to the x-axis, so

$$\mathbf{N}_s = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & 1/\lambda_1^2 \end{bmatrix} \quad (29)$$

$$= \lambda_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1/\lambda_1^2 - \lambda_1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (30)$$

$$= \lambda_1\mathbf{g} + (1/\lambda_1^2 - \lambda_1)\mathbf{cc}, \quad (31)$$

where the last expression is the coordinate-independent form, **g** is the unit tensor and **cc** is the 'outer product' of **c**, given by the last matrix term in expression (30).

If we substitute (31) in (12), we have

$$w(\lambda_1\mathbf{g} + (1/\lambda_1^2 - \lambda_1)\mathbf{cc})\mathbf{b}_f = \mathbf{K}. \quad (32)$$

We have

$$(\mathbf{cc})\mathbf{b}_f = \mathbf{c}(\mathbf{c}\cdot\mathbf{b}_f) \quad (33)$$

(Malvern 1969, p. 36) so (32) simplifies to

$$w\lambda_1\mathbf{b}_f + w(1/\lambda_1^2 - \lambda_1)(\mathbf{c}\cdot\mathbf{b}_f)\mathbf{c} = \mathbf{K}. \quad (34)$$

This tells us that **K** is formed by adding multiples of the vectors $\mathbf{b}_f$ and **c**. Therefore these vectors must be coplanar. Let **v** be the unit vector parallel to **K**. A useful measure of the deviation from the situation in which the constraints are satisfied is given by

$$\mu = \mathbf{v}\cdot(\mathbf{b}_f \times \mathbf{c}) \quad (35)$$

so the constraint is

$$\mu = 0. \quad (36)$$

Ideally $\mu = 0$; it can range up to ± 1. For the moment assume that the input data satisfies the constraint. The two unknowns in equation (34) are w and λ_1. By dot-multiplying the expression with **c**, one obtains

$$w(\mathbf{c}\cdot\mathbf{b}_f)/\lambda_1^2 = \mathbf{c}\cdot\mathbf{K} \quad (37)$$

and by dot-multiplying (34) with $\mathbf{b}_f$,

$$w(\lambda_1 + (1/\lambda_1^2 - \lambda_1)(\mathbf{c}\cdot\mathbf{b}_f)^2) = \mathbf{b}_f\cdot\mathbf{K}. \quad (38)$$

w can be eliminated from (37) and (38), and the resulting expression solved for λ_1 to give

$$\lambda_1^3 = \frac{\mathbf{c}\cdot\mathbf{b}_f(\mathbf{b}_f\cdot\mathbf{K} - (\mathbf{c}\cdot\mathbf{b}_f)(\mathbf{c}\cdot\mathbf{K}))}{\mathbf{c}\cdot\mathbf{K}(1 - (\mathbf{c}\cdot\mathbf{b}_f)^2)}. \quad (39)$$

Substituting the definitions of ψ and ω, (39) reduces to

$$\lambda_1^3 = \frac{\tan\psi}{\tan\omega}. \quad (40)$$

Given λ_1, the irrotational part of the deformation is

$$\mathbf{D} = \lambda_1^{1/2}\mathbf{g} + (1/\lambda_1 - \lambda_1^{1/2})\mathbf{cc} \quad (41)$$

which is derived in an analogous way to (31). This can be substituted in (5) and (6) to give the initial fabric and bedding.

As noted in the main text, the data must be adjusted so as to satisfy the constraint before the factorization is performed. The correction should be made to $\mathbf{N}_f^*$ and chosen so that it vanishes when $\mu = 0$ and is algebraically simple. A convenient correction is

$$\mathbf{N}_f^c \propto \mathbf{N}_f^* - (\mathbf{K}\cdot\mathbf{m})(\mathbf{b}_f\mathbf{m} + \mathbf{m}\mathbf{b}_f) \quad (42)$$

where the proportionality sign shows that $\mathbf{N}_f^c$ should be renormalized to unit determinant. **m** is the unit vector parallel to $\mathbf{b}_f \times \mathbf{c}$, the bedding/cleavage intersection. The corrected version should be used all the way through the analysis.

APPENDIX 4: 3D STRAIN ANALYSIS ASSUMING UNIAXIAL INITIAL FABRIC ELLIPSOID

To analyse the situation in which the initial fabric ellipsoid is assumed uniaxial, eqn (12), though it is still true, cannot be used as a starting point since it does not refer directly to the initial fabric ellipsoid. By analogy with (31) we can write

$$\mathbf{N}_i^* = u\mathbf{g} + (1/u^2 - u)\mathbf{b}_i\mathbf{b}_i, \quad (43)$$

where u is the squared length of the bedding-parallel radius of the fabric ellipsoid. Substitute this into the right-hand side of (3) to find

$$\mathbf{N}_f^* = \mathbf{D}[u\mathbf{g} + (1/u^2 - u)\,\mathbf{b}_i\mathbf{b}_i]\mathbf{D}^T \quad (44)$$

$$= u\mathbf{N}_s + (1/u^2 - u)(\mathbf{D}\mathbf{b}_i)(\mathbf{D}\mathbf{b}_i). \quad (45)$$

Using (6) we find

$$\mathbf{D}\mathbf{b}_i = \mathbf{N}_s\mathbf{b}_f(\mathbf{b}_f\cdot\mathbf{N}_s\mathbf{b}_f)^{-1/2} \quad (46)$$

$$= (w\mathbf{b}_f\cdot\mathbf{K})^{-1/2}\mathbf{K} \quad (47)$$

using equation (12). Substituting this in (45), and noting that because $\det \mathbf{N} = 1$, $u^2w = 1$, find

$$\mathbf{N}_f^* = u\mathbf{N}_s + ((1 - u^3)/\mathbf{b}_f\cdot\mathbf{K})\mathbf{KK}. \quad (48)$$

It may be checked, by dot-multiplying both sides of (48) with $\mathbf{b}_f$, that eqn (12) can be recovered. Instead dot-multiply both sides with **c**

$$\mathbf{N}_f^*\mathbf{c} = u\lambda_3\mathbf{c} + ((1-u^3)(\mathbf{c}\cdot\mathbf{K})/(\mathbf{b}_f\cdot\mathbf{K}))\mathbf{K}, \tag{49}$$

where we have assumed the cleavage is a principal plane of the strain ellipsoid, and invoked (23). Let

$$\mathbf{Q} = \mathbf{N}_f^*\mathbf{c} \tag{50}$$

so **Q** is analogous to **K**. **Q** is parallel to the line joining the centre of the final fabric ellipsoid to the point where the cleavage plane is tangent to that ellipsoid. (49) is analogous to (35). So the constraint in this case is that the three vectors **Q**, **c** and **K** should be coplanar. Let **t** be the unit vector parallel to **Q**, so the constraint is that the unit vectors **c**, **t** and **v** should be coplanar. Let

$$\nu = \mathbf{c}\cdot(\mathbf{t}\times\mathbf{v}) \tag{51}$$

then

$$\nu = 0 \tag{52}$$

is the constraint. It is shown in Appendix 5 that the constraint (52) is equivalent to

$$\mathbf{c}\cdot(\mathbf{N}_f^*)^{-1}\mathbf{m} = 0 \tag{53}$$

and this suggests that we can correct $\mathbf{N}_f^*$ by

$$\mathbf{N}_f^c \propto [\mathbf{N}_f^* - (\mathbf{c}\cdot(\mathbf{N}_f^*)^{-1}\mathbf{m})(\mathbf{mc}+\mathbf{cm})]^{-1} \tag{54}$$

Equation (49) contains two unknowns, u and λ_3. By dot-multiplying (49) with **c** one obtains

$$\mathbf{c}\cdot\mathbf{Q} = u\lambda_3 + (1-u^3)(\mathbf{c}\cdot\mathbf{K})^2/(\mathbf{b}_f\cdot\mathbf{K}) \tag{55}$$

and by dotting (49) with $\mathbf{b}_f$,

$$\mathbf{K}\cdot\mathbf{c} = u\lambda_3\mathbf{c}\cdot\mathbf{b}_f + (1-u^3)(\mathbf{c}\cdot\mathbf{K}). \tag{56}$$

Note that $\mathbf{K}\cdot\mathbf{c} = \mathbf{Q}\cdot\mathbf{b}_f$. By eliminating λ_3 from (55) and (56),

$$u^3 = \frac{(\mathbf{c}\cdot\mathbf{b}_f)((\mathbf{c}\cdot\mathbf{Q})(\mathbf{b}_f\cdot\mathbf{K}) - (\mathbf{c}\cdot\mathbf{K})^2)}{(\mathbf{c}\cdot\mathbf{K})(\mathbf{b}_f\cdot\mathbf{K} - (\mathbf{c}\cdot\mathbf{b}_f)(\mathbf{c}\cdot\mathbf{K}))}. \tag{57}$$

This is more complex than (39). The expression for u involves four vectors $\mathbf{b}_f$, **c**, **Q** and **K**, only the last three of which are coplanar. Therefore it cannot easily be reexpressed in a form referring to angles measured in a single plane. To obtain the strain ellipsoid, note that (48) can be written

$$\mathbf{N}_s = (1/u)\mathbf{N}_f^* + ((u^2 - 1/u)/(\mathbf{b}_f\cdot\mathbf{K})\mathbf{KK}. \tag{58}$$

When $\mathbf{N}_s$ is found, its eigenvectors and eigenvalues are calculated by standard methods (e.g. Ramsay 1967, p. 142) and the irrotational part of the deformation is defined. Using this in eqns (5) and (6) gives an initial uniaxial ellipsoid, with u as one eigenvalue.

APPENDIX 5: THE EQUIVALENCE OF EXPRESSIONS (52) AND (53)

The proof that the constraint (52) is equivalent to (53) is as follows. We have

$$\nu = \mathbf{c}\cdot(\mathbf{t}\times\mathbf{v}) \tag{59}$$

$$\propto \mathbf{cp} \tag{60}$$

where we have defined

$$\mathbf{p} = (\mathbf{Nb}_f)\times(\mathbf{Nc}).$$

From the definition of the vector cross-product, note that **p** is orthogonal to $\mathbf{Nb}_f$, **Nc** and any arbitrary linear combination of those two vectors. Therefore

$$\mathbf{p}\cdot(\alpha\mathbf{Nb}_f + \beta\mathbf{Nc}) = 0 \tag{61}$$

for any values of α and β. Since **N** is symmetrical this can be rewritten as

$$(\mathbf{Np})\cdot(\alpha\mathbf{b}_f + \beta\mathbf{c}) = 0. \tag{62}$$

So **Np** is a vector which is orthogonal to any linear combination of $\mathbf{b}_f$ and **c**. Any such vector must be proportional to the cross-product of $\mathbf{b}_f$ and **c**, and therefore to the unit vector parallel to bedding/cleavage intersection, **m**. Thus

$$\mathbf{Np} \propto \mathbf{m} \tag{63}$$

and

$$\mathbf{p} \propto \mathbf{N}^{-1}\mathbf{m}. \tag{64}$$

Finally, putting $\mathbf{M} = \mathbf{N}^{-1}$ and using (60), we find $\nu = 0$ implies

$$\mathbf{c}\cdot\mathbf{Mm} = 0$$

and the result is proved.